Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MTH101A, End Semester Examination Grading Scheme
April 30, 2013

Marks: 100
Time: 3 Hours

***Answer all questions. A*ll the parts of each question must be answered in continuation; otherwise they will not be graded.**

1. (a) Let the sequence $\{x_n\}_{n=2}^{\infty}$ of real numbers be defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right);\ n = 1,2,\ldots\ and\ a, x_1 > 0.$$

Prove that (i) $\{x_n\}_{n=2}^{\infty}$ is bounded below by $\sqrt{a}$ and (ii) $\{x_n\}_{n=2}^{\infty}$ is non-increasing. *(6)*

Solution: (i) $2x_{n+1}x_n = x_n^2 + a \geq 2\sqrt{x_n^2 a}$ $(by\ AM - GM\ inequality)$ *(2 marks)*

$\Rightarrow x_{n+1} \geq \sqrt{a}$ for $n = 1,2,\ldots$ (since $x_n > 0$) *(1 mark)*

(ii) $x_{n+1} - x_n = \frac{1}{2}(x_n + \frac{a}{x_n}) - x_n = \frac{1}{2}(\frac{a - x_n^2}{x_n}) \leq 0$ (by (i)) *(3 marks)*

(b) Prove that $P(x) = 2x^{13} + 13x^2 + 26x + 6$ has exactly one real root. *(6)*

Solution: $P(0) > 0,\ P(-1) < 0 \Rightarrow P(x)$ has at least one real root (by IVT) (2 marks)

The function $P'(x) = 26x^{12} + 26x + 26 > 0$ for every real x, since it has a positive minima at $x = -\frac{1}{12^{1/11}}$. (2 marks)

$\Rightarrow P(x)$ can have at most one root (by MVT) (1 marks)

$\Rightarrow P(x)$ has exactly one real root. (1 mark)

2. (a) Use Comparison Test to determine whether the improper integral $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx$ converges or diverges. **(7)**

Solution: $\int_0^\infty \frac{1}{x^2+\sqrt{x}}dx = \int_0^1 \frac{1}{x^2+\sqrt{x}}dx + \int_1^\infty \frac{1}{x^2+\sqrt{x}}dx \equiv I_1 + I_2$ (1 mark)

$\frac{1}{x^2+\sqrt{x}} \le \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{1}{\sqrt{x}}dx$ converges $\Rightarrow I_1$ converges (by Comparison Test) (3 marks)

$\frac{1}{x^2+\sqrt{x}} \le \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}dx$ converges $\Rightarrow I_2$ converges (by Comparison Test) (3 marks)

(b) Use Limit Comparison Test to determine whether the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$ converges or diverges. **(6)**

Solution: $\lim_{n\to\infty} \frac{n}{n^{1+(1/n)}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (3+2 marks)

Therefore, by Limit Comparison Test $\sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$ diverges. (1 mark)

3. (a) Using Theorem of Pappus, determine the volume of solid generated by revolving an equilateral triangle of side-length a, with its base lying on $x-axis$, about the line L parallel to its base and intersecting $y-axis$ at the point $(0,-c)$, $c > a$. ***(7)***

Solution: By Pappus Theorem,

Volume of solid generated = Area of Triangle$\times 2\pi$ (Distance of Centroid of Triangle from L) (2 marks)

$$= \left(\frac{1}{2}a \times a\sin\frac{\pi}{3}\right)\times 2\pi\left(c+\frac{a\sin\frac{\pi}{3}}{3}\right) = \frac{\pi a^2\sqrt{3}}{2}\left(c+\frac{a\sqrt{3}}{6}\right)$$ (3+2 marks)

(b) A curved wedge is cut from a cylinder of radius 5 by two planes. One plane is perpendicular to the axis of cylinder. The second plane crosses the first plane at a 60° angle at the center of the cylinder. Find the volume of the wedge. **(6)**

Solution: Cross-section of wedge by a plane passing through $(x,0)$ and perpendicular to $x-axis$ is a rectangle with width $2\sqrt{25-x^2}$ and height $x\tan\frac{\pi}{3} = x\sqrt{3}$. (3 marks)

$Area\ of\ cross\text{-}section = \sqrt{3}\,x\sqrt{25-x^2}$ (1 mark)

Therefore, Required Volume $= 2\sqrt{3}\int_0^5 x\sqrt{25-x^2}\,dx$ (2 marks)

4. (a) Find the equation of line which passes through $(0, \frac{1}{3}, \frac{3}{4})$ and lies in both the planes $2x - 4z + 3 = 0, 4x - 3y + 1 = 0$. ***(7)***

Solution: Let $\vec{N}_1, \vec{N}_2$ be the normals of the given planes. The required line is parallel to

$$\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} i & j & k \\ 2 & 0 & -4 \\ 4 & -3 & 0 \end{vmatrix} = -12\hat{i} - 16\hat{j} - 6\hat{k} .$$ (4 marks)

Since this line passes through $(0, \frac{1}{3}, \frac{3}{4})$, its equation is

$$\frac{x-0}{-12} = \frac{y - \frac{1}{3}}{-16} = \frac{z - \frac{3}{4}}{-6} .$$ (3 marks)

(b) Reparametrize the curve $\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + \sqrt{5}\,t\,\hat{k},\ 0 \le t \le 2\pi$, in terms of arc-length. Also determine unit tangent vector and unit normal vectors of this curve in terms of arc-length parameter. ***(6)***

Solution: $s(t) = \int_0^t \sqrt{4\sin^2 t + 4\cos^2 t + 5} = 3t$

$\Rightarrow \vec{r}(s) = (2\cos\frac{s}{3})\hat{i} + (2\sin\frac{s}{3})\hat{j} + (\sqrt{5}\frac{s}{3})\hat{k}$ is the required parametric equation in terms of arc-length. (2 marks)

Unit Tangent Vector in terms of arc-length is given by

$$\vec{T}(s) = \vec{r}'(s) = (-\frac{2}{3}\sin\frac{s}{3})\hat{i} + (\frac{2}{3}\cos\frac{s}{3})\hat{j} + (\frac{\sqrt{5}}{3})\hat{k}$$ (2 marks)

Unit Normal Vectors in terms of arc-length are given by

$$\hat{n}(s) = \pm\frac{\vec{T}'(s)}{\left|\vec{T}'(s)\right|} = \mp[(\cos\frac{s}{3})\hat{i} + (\sin\frac{s}{3})\hat{j}]$$ (2 marks)

Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MTH101A, End Semester Examination Grading Scheme
April 30, 2013

Marks: 100
Time: 3 Hours

***Answer all questions.* All the parts of each question must be answered in continuation; otherwise they will not be graded.**

5. (a) Using Lagrange Multiplier Method, maximize the function $f(x,y,z)=xyz$ subject to the constraint $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where a, b and c are positive constants. ***(7)***

Solution: Let $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$\vec{\nabla} f=\lambda\vec{\nabla} g \Rightarrow yz=\frac{2\lambda x}{a^2}, xz=\frac{2\lambda y}{b^2}, yx=\frac{2\lambda z}{c^2}$ (2 marks)

$\Rightarrow 3xyz=2\lambda \Rightarrow \lambda=\frac{3}{2}xyz$ (2 marks)

Therefore, $yz=\frac{2\lambda x}{a^2}\Rightarrow yz=\frac{3x^2yz}{a^2}\Rightarrow x=\pm\frac{a}{\sqrt{3}}$. Similarly, $y=\pm\frac{b}{\sqrt{3}}, z=\pm\frac{c}{\sqrt{3}}$ (2 marks)

$\Rightarrow \max f(x,y,z)=\frac{abc}{3\sqrt{3}}$ (1 mark)

(b) Express the surface area of portion of sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=2x$ in the form of a double integral with suitable limits. ***(6)***

Solution: Surface area of the sphere = 2 surface area of the graph of $z=f(x,y)\equiv\sqrt{4-x^2-y^2}$

Required surface area $=2\iint_T \sqrt{1+f_x^2+f_y^2}\; dx\,dy$, where T is the projection of required surface area on $xy-plane$. (2 marks)

$\Rightarrow$ Required surface area $=2\iint_T \frac{2}{\sqrt{4-x^2-y^2}}dxdy$ (2 marks)

$=2\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\frac{2}{\sqrt{4-r^2}}r\,drd\theta$ (2 marks)

6. (a) Find equation of the surface generated by the normals to the surface $x+3yz+2xyz^2=0$ at all points of $y-axis$. (6)

Solution: The normal to given surface is $\vec{N}=(1+2yz^2, 3z+2xz^2, 3y+4xyz)$ (2 marks)

$\Rightarrow \vec{N}=(1,0,3t)$ at an arbitrary point $(0,t,0)$ of $y-axis$ (1 marks)

$\Rightarrow$ For any point (x,y,z) on the required surface, $\frac{x}{1}=\frac{z}{3t},\ y=t$ (2 marks)

$\Rightarrow$ Equation of Required surface: $z=3xy$. (1 marks)

(b) Let $f(x,y)=\frac{y}{|y|}\sqrt{x^2+y^2},\ if\ \ y\neq 0\ \ and\ \ f(x,0)=0$. **Show that (i)** $f(x,y)$ **has all directional derivatives at** $(0,0)$ **(ii)** $f(x,y)$ **is not differentiable at** $(0,0)$**. (6)**

Solution: For $\|(u_1,u_2)\|=1$, $\lim_{t\to 0}\frac{f(tu_1,tu_2)}{t}=0,\ if\ \ u_2=0\ \ and \lim_{t\to 0}\frac{f(tu_1,tu_2)}{t}=\frac{u_2}{|u_2|}, if\ u_2\neq 0$.

$\Rightarrow f(x,y)$ has all directional derivatives at $(0,0)$ (3 marks)

Since $f_x(0,0)=0,\ f_y(0,0)=1$, with $\alpha=(0,1)$, let

$$\varepsilon(h,k)=\frac{f(h,k)-f(0,0)-\alpha.(h,k)}{\|(h,k)\|}=\frac{\frac{k}{|k|}\sqrt{h^2+k^2}-k}{\sqrt{h^2+k^2}}$$ (1 mark)

Since $\varepsilon(k,k)=\frac{(\sqrt{2}-1)k}{\sqrt{2}\ |k|}$ does not tend to 0 as $k\to 0$, $\varepsilon(h,k)$ does not tend to 0 as $(h,k)\to(0,0)$. Therefore $f(x,y)$ is not differentiable at $(0,0)$. (2 marks)

7. (a) Determine the points of maxima, minima and saddle points for the function $f(x,y)=4xy-x^4-y^4$. (6)

Solution: $f_x=4y-4x^3, f_y=4x-4y^3\ so\ that\ f_x=f_y=0\Rightarrow(x,y)=(0,0),\ (1,1)\ or\ (-1,-1)$. (2 marks)

$f_{xx}=-12x^2, f_{yy}=-12y^2,\ f_{xy}=4,\ D=f_{xx}f_{yy}-(f_{xy})^2=12\times 12x^2y^2-16$ (1 mark)

$\Rightarrow D>0\ and\ \ f_{xx}<0\ at\ (1,1)\ and\ (-1,-1)\ \Rightarrow f(x,y)$ has maxima at $(1,1)$ and $(-1,-1)$ (2 marks)

$D<0\ at\ (0,0)\Rightarrow f(x,y)$ has saddle point at $(0,0)$. (1 mark)

(b) Using Green's theorem evaluate $\oint_L(-y\sec^2(x-1)\hat{i}+(y^2+1)\hat{j}).d\vec{R}$, where L is the square with vertices at $(0,0),(2,0),(2,2),(0,2)$ described counter clockwise . *(6)*

Solution.

$$\oint_C\vec{F}\cdot d\vec{R}=\iint_D(Curl\,\vec{F}.\hat{k})\ dxdy=\int_0^2\int_0^2\sec^2(x-1)\ dxdy$$ (2+2 marks)

$$=(2\tan 1)\int_0^2 dy=4\tan 1$$ (2 marks)

8. (a) State Stoke's Theorem precisely. Using this theorem evaluate, $\oint_C \vec{\nabla} f . d\vec{R}$**,**

where $f(x,y,z) = \sin x - 3\cos y + z^5$ **and** $C : 3x^2 + 2y^2 = 6,\ z = 2$ **is the curve oriented counter clockwise.**

(6)

Solution: Statement of Stoke's Theorem (1 mark)

$curl\, \vec{\nabla} f = 0$ (2 marks)

$\oint_C \vec{\nabla} f \cdot d\vec{r} = \iint_S Curl\, \vec{\nabla} f \cdot \hat{n}\ dS = 0$ (3 marks)

(b) Let *G* be the domain bounded by the hemisphere $x^2 + (y-2)^2 + z^2 = 25$ **and the plane** $y = 2$**. Let** $\vec{F}(x,y,z) = x\,\hat{i} + (y-2)\,\hat{j} + 5z\,\hat{k}$**. Use Gauss Divergence Theorem to evaluate** $\iint_S \vec{F}.\hat{n}\ d\sigma$**, where** $\hat{n}$ **is unit outward normal to the bounding surface** S **of domain** G**. (6)**

Solution:

By Gauss Divergence Theorem ,

$\iint_S \vec{F}.\hat{n}\ d\sigma = \iiint_G Div\vec{F}\, dV = \iiint_G 7\, dV$ (2 marks)

$= 7 \times$ Volume of the Hemisphere (2 marks)

$= 7 \times \frac{2}{3} \pi \times 5^3$ (2 marks)

.